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Generators of Dynamical Semigroups

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We give a detailed description of the generators of those strongly continuous quantum dynamical semigroups which possess a pure stationary state or an associated extension property.

1. INTRODUCTION

If $\mathscr{T}(\mathscr{H})$ is the Banach space of trace class operators on a Hilbert space $\mathscr{H}$, we define a dynamical semigroup T_t on $\mathscr{T}(\mathscr{H})$ to be a strongly continuous one-parameter semigroup of completely positive contractions on $\mathscr{T}(\mathscr{H})$. The adjoint operators T_t^* on $\mathscr{L}(\mathscr{H})$ define an ultraweakly continuous one-parameter semigroup of normal completely positive contractions on $\mathscr{L}(\mathscr{H})$, and every such semigroup on $\mathscr{L}(\mathscr{H})$ is the adjoint of a dynamical semigroup on $\mathscr{T}(\mathscr{H})$.

If T_t is norm continuous a complete analysis of the form of its (bounded) infinitesimal generator W was given by Lindblad [14]; in fact he studied W^* but this is only a matter of technical convenience. In this paper we make substantial progress toward the classification of the generators W of strongly continuous dynamical semigroups T_t, under the assumption that T_t possesses a pure stationary state, that is,

$$T_t\{|\Omega\rangle\langle\Omega|\} = |\Omega\rangle\langle\Omega| \tag{1.1}$$

for some unit vector $\Omega \in \mathscr{H}$ and all $t \geqslant 0$. Equivalently we assume that $|\Omega\rangle\langle\Omega|$ lies in Dom (W) and that

$$W\{|\Omega\rangle\langle\Omega|\} = 0. \tag{1.2}$$

The physical necessity for such an assumption is unclear and in Section 4 we replace it by an extension property, which may possibly always be satisfied.

We follow as far as possible the notation of [3], to which this paper is complementary. A proof of the following proposition of Ingarden and Kossakowski [10] may be extracted from Theorem 3 of [7]. See also [8, 11].

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PROPOSITION 1.1. *There exists a strongly continuous one-parameter contraction semigroup C_t on $\mathscr{H}$ such that*

$$C_t^*(X\Omega) = (T_t^*X)\Omega \tag{1.3}$$

for all $X \in \mathscr{L}(\mathscr{H})$ and $t \geqslant 0$.

We comment that the strong continuity of C_t, its weak continuity, and the the strong continuity of C_t^* are equivalent by [2, p. 103]. If the infinitesimal generator of C_t is Y, then that of C_t^* is Y^*. We define the dynamical semigroup S_t on $\mathscr{T}(\mathscr{H})$ by

$$S_t\rho = C_t\rho C_t^* \tag{1.4}$$

for all $\rho \in \mathscr{T}(\mathscr{H})$ and $t \geqslant 0$.

LEMMA 1.2. *The vector Ω lies in the domains of Y and Y^*, and*

$$Y\Omega = Y^*\Omega = 0. \tag{1.5}$$

Proof. If $t > 0$, then

$$\begin{aligned} \langle C_t^*\Omega, \Omega\rangle &= \langle T_t^*(1)\Omega, \Omega\rangle \\ &= \mathrm{Tr}[T_t^*(1) \mid \Omega\rangle\langle\Omega \mid] \\ &= \mathrm{tr}[T_t\{\mid \Omega\rangle\langle\Omega \mid\}] \\ &= 1. \end{aligned}$$

Since C_t is a contraction

$$C_t\Omega = C_t^*\Omega = \Omega$$

for all $t > 0$. Differentiation yields Eq. (1.5).

2. THE NORM CONTINUOUS CASE

In this section we suppose that T_t is a dynamical semigroup on $\mathscr{T}(\mathscr{H})$ satisfying the following hypothesis:

I. T_t is norm continuous and possesses a pure stationary state $\mid \Omega\rangle\langle\Omega \mid$. Although our results below can be derived from Lindblad's theorem [14], we follow a method which extends to the strongly continuous case, which is treated in Section 3.

LEMMA 2.1. *Under hypothesis* I *the operators* Y *and* Y^* *are bounded.*

Proof. If $\psi \in \mathscr{H}$ and $X = |\psi\rangle\langle\Omega|$, then

$$\begin{aligned} Y^*\Omega &= \lim_{t\to 0} t^{-1}(C_t^* - 1)\, X\Omega \\ &= \lim_{t\to 0} \{t^{-1}(T_t^* X - X)\}\Omega \\ &= (W^* X)\Omega. \end{aligned}$$

Since Y^* is closed and has domain equal to $\mathscr{H}$, it is bounded.

LEMMA 2.2. *The bounded linear map* J *on* $\mathscr{T}(\mathscr{H})$ *defined by*

$$W(\rho) = Y\rho + \rho Y^* + J(\rho) \tag{2.1}$$

is completely positive.

Proof. If $X_0, X_1 \in \mathscr{L}(\mathscr{H})$ and X_1 is invertible, then

$$T_t^*(X_0^* X_0) \geqslant T_t^*(X_0^* X_1)\, T_t^*(X_1^* X_1)^{-1}\, T_t^*(X_1^* X_0)$$

for all $t \geqslant 0$ by [6, 13]. Differentiating at $t = 0$ and replacing X_0 by $X_1 X_0$ yields

$$\begin{aligned} &W^*(X_0^* X_1^* X_1 X_0) + X_0^* W^*(X_1^* X_1)\, X_0 \\ &\qquad \geqslant W^*(X_0^* X_1^* X_1)\, X_0 + X_0^* W^*(X_1^* X_1 X_0). \end{aligned}$$

Using Eq. (1.2) we now obtain

$$\begin{aligned} &\langle W^*(X_1^* X_1)\, X_0\Omega, X_0\Omega\rangle \\ &\qquad \geqslant \langle X_0\Omega, Y^* X_1^* X_1 X_0\Omega\rangle + \langle Y^* X_1^* X_1 X_0\Omega, X_0\Omega\rangle. \end{aligned}$$

Putting $\psi = X_0\Omega$ and $A = X_1^* X_1$ yields

$$\langle W^*(A)\psi, \psi\rangle \geqslant \langle \psi, Y^* A\psi\rangle + \langle Y^* A\psi, \psi\rangle$$

for all $\psi \in \mathscr{H}$ so that

$$W^*(A) \geqslant AY + Y^* A$$

for all invertible $A \geqslant 0$ and hence all $A \geqslant 0$. Thus J^* and hence J are positive. Complete positivity follows in the usual manner.

THEOREM 2.3. *If T_t satisfies hypothesis I, its generator W is of the form*

$$W(\rho) = Y\rho + \rho Y^* + \sum_{n=1}^{\infty} B_n \rho B_n^*, \tag{2.2}$$

where

$$Y\Omega = Y^*\Omega = B_n\Omega = 0$$

for all n and

$$Y + Y^* + \sum_{n=1}^{\infty} B_n^* B_n \leqslant 0, \tag{2.3}$$

the sum converging in the strong operator topology.

Proof. Since J^* is normal and completely positive, it has the form

$$J^*(X) = \sum_{n=1}^{\infty} B_n^* X B_n, \tag{2.4}$$

where $B_n \in \mathscr{L}(\mathscr{H})$ and $\sum B_n B_n^*$ is strongly convergent [2, p. 140; 12]. This yields Eq. (2.2). By Eqs. (1.2) and (1.5)

$$\sum_{n=1}^{\infty} B_n \mid \Omega \rangle\langle \Omega \mid B_n^* = 0$$

so $B_n\Omega = 0$ for all n. Finally since

$$T_t^*(1) \leqslant 1$$

for all $t \geqslant 0$

$$W^*(1) \leqslant 0$$

which is equivalent to Eq. (2.3).

The following corollary will be used in Section 3.

COROLLARY 2.4. *If T_t satisfies hypothesis I and π: $\mathscr{T}(\mathscr{H}) \to \mathscr{T}(\mathscr{H})$ is defined by*

$$\pi(\rho) = (Y - 1)^{-1} \rho (Y^* - 1)^{-1}, \tag{2.5}$$

then

$$\| T_t \pi\rho - S_t \pi\rho \| \leqslant 4t \| \rho \| \tag{2.6}$$

for all $\rho \in \mathscr{T}(\mathscr{H})$ *and* $t \geqslant 0$. *Moreover*

$$T_t\rho \geqslant S_t\rho \geqslant 0 \tag{2.7}$$

for all $\rho \in \mathscr{T}(\mathscr{H})^+$ *and* $t \geqslant 0$.

Proof. The positive linear map $J_1 = J\pi$ on $\mathscr{T}(\mathscr{H})$ has norm

$$\begin{aligned}\| J_1 \| &= \| J_1^* \| = \| J_1^*(1) \| \\ &= \left\| (Y-1)^{-1} \left(\sum B_n^* B_n\right) (Y^*-1)^{-1} \right\| \\ &\leqslant \|(Y-1)^{-1}(Y+Y^*)(Y^*-1)^{-1}\|\end{aligned}$$

by Eq. (2.3). The estimate

$$\|(Y-1)^{-1}\| = \left\| \int_0^\infty C_t e^{-t}\, dt \right\| \leqslant 1$$

combined with simple algebra yields

$$\| J_1 \| \leqslant 4.$$

Now the perturbation theory of semigroups [2, p. 9] implies that

$$T_t = S_t + \int_{s=0}^{t} T_{t-s} J S_s\, ds \tag{2.8}$$

so

$$\begin{aligned}\| T_t\pi\rho - S_t\pi\rho \| &\leqslant \int_{s=0}^{t} \| T_{t-s} J S_s \pi\rho \|\, ds \\ &= \int_{s=0}^{t} \| T_{t-s} J_1 S_s \rho \|\, ds \\ &\leqslant 4t \|\rho\|\end{aligned}$$

as required. Equation (2.7) is an immediate consequence of Eq. (2.8) and Lemma 2.2.

3. The Strongly Continuous Case

In this section we suppose that T_t is a dynamical semigroup on $\mathscr{T}(\mathscr{H})$ satisfying the following hypothesis.

II. T_j has a pure stationary state $|\Omega\rangle\langle\Omega|$ and satisfies

$$T_t^*\{\mathscr{C}(\mathscr{H})\} \subseteq \mathscr{C}(\mathscr{H})$$

for all $t \geqslant 0$, where $\mathscr{C}(\mathscr{H})$ is the set of compact operators on $\mathscr{H}$.

LEMMA 3.1. *Under hypothsis* II *one has*

$$\| T_t\pi\rho - S_t\pi\rho \| \leqslant 4t \| \rho \| \tag{3.1}$$

for all $\rho \in \mathscr{T}(\mathscr{H})$ *and* $t > 0$. *Moreover*

$$T_t\rho \geqslant S_t\rho \geqslant 0 \tag{3.2}$$

for all $\rho \in \mathscr{T}(\mathscr{H})^+$ *and* $t \geqslant 0$.

Proof. Given $\epsilon > 0$ we let T_t^ϵ be the norm continuous dynamical semigroup on $\mathscr{T}(\mathscr{H})$ given by

$$T_t^\epsilon\rho = e^{-\epsilon^{-1}t} \sum_{n=0}^{\infty} (n!)^{-1}\, \epsilon^{-n}t^n T_{\epsilon n}\rho$$

with the bounded generator

$$W^\epsilon = \epsilon^{-1}(T_\epsilon - 1).$$

Noting that $|\Omega\rangle\langle\Omega|$ is a stationary state for T_t^ϵ, we label all the associated operators constructed by the method of Section 2 with an ϵ. Note in particular that

$$C_t^\epsilon = e^{-\epsilon^{-1}t} \sum_{n=0}^{\infty} (n!)^{-1}\, \epsilon^{-n}t^n C_{\epsilon n}\,.$$

If $\rho \in \mathscr{T}(\mathscr{H})$ and we define

$$\sigma_t^\epsilon = T_t^\epsilon\pi^\epsilon\rho - S_t^\epsilon\pi^\epsilon\rho$$

and

$$\sigma_t = T_t\pi\rho - S_t\pi\rho,$$

then

$$\lim_{\epsilon\to 0} \| \sigma_t^\epsilon - \sigma_t \| = 0$$

so

$$\| \sigma_t \| \leqslant 4t \| \rho \|$$

by Corollary 2.4 Equation (3.2) is deduced similarly from Eq. (2.7).

In order to prove Lemma 3.3 we need the following proposition, which is a small variation of Theorem 10.7.2 of [9]. See also [5].

PROPOSITION 3.2. *If T_t is a strongly continuous one-parameter semigroup of weak*-continuous contractions on a Banach dual space $\mathscr{B}^*$, then the domain of the infinitesimal generator W is*

$$\mathscr{D}_0 = \{\rho \in \mathscr{B}^*: \liminf_{t\to 0} t^{-1} \| T_t\rho - \rho \| < \infty\}.$$

We return now to the problem at hand, putting $\mathscr{B} = \mathscr{C}(\mathscr{H})$ so that $\mathscr{B}^*$ may be identified with $\mathscr{T}(\mathscr{H})$.

LEMMA 3.3. *The domain of W contains*

$$\mathscr{D} = \pi(\mathscr{T}(\mathscr{H})\}. \tag{3.3}$$

Proof. Hypothesis II implies that each operator T_t is weak* continuous so we need only prove that $\mathscr{D} \subseteq \mathscr{D}_0$. By inspection $\mathscr{D}$ is contained in the domain of the infinitesimal generator Z of S_t which is formally

$$Z\rho = Y\rho + \rho Y^*$$

so

$$\liminf_{t\to 0} t^{-1} \| S_t\rho - \rho \| < \infty$$

for all $\rho \in \mathscr{D}$. By Lemma 3.1 if $\rho \in \mathscr{D}$, then

$$t^{-1} \| T_t\rho - \rho \| \leqslant t^{-1} \| S_t\rho - \rho \| + 4 \| \pi^{-1}\rho \|$$

so

$$\liminf_{t\to 0} t^{-1} \| T_t\rho - \rho \| < \infty.$$

THEOREM 3.4. *If T_t satisfies hypothesis* II, *then*

$$W\rho = Y\rho + \rho Y^* + \sum_{n=1}^{\infty} B_n\rho B_n^* \tag{3.4}$$

for all $\rho \in \mathscr{D}$, where B_n are linear maps from Dom(Y) *to $\mathscr{H}$ such that*

$$Y\Omega = Y^*\Omega = B_n\Omega = 0 \tag{3.5}$$

for all n. Moreover

$$Y + Y^* + \sum_{n=1}^{\infty} B_n^*B_n \leqslant 0 \tag{3.6}$$

in the sense that

$$\langle Y\phi, \phi\rangle + \langle \phi, Y\phi\rangle + \sum_{n=1}^{\infty} \| B_n \phi \|^2 \leqslant 0 \tag{3.7}$$

for all $\phi \in \mathrm{Dom}(Y)$.

Proof. If $\rho \in \mathscr{T}(\mathscr{H})$, then

$$\begin{aligned} J_1(\rho) &= \lim_{t\to 0} t^{-1}(T_t \pi\rho - S_t \pi\rho) \\ &= \lim_{t\to 0} t^{-1}(T_t\pi\rho - \pi\rho) - \lim_{t\to 0} t^{-1}(S_t\pi\rho - \pi\rho) \\ &= W\pi\rho - Y(\pi\rho) - (\pi\rho)\, Y^*. \end{aligned}$$

The first line of this equation and Eq. (3.2) imply that J_1 is positive, and hence bounded by [2, p. 17]. The complete positivity of J_1 follows in the usual manner. Since J_1^* is normal and completely positive on $\mathscr{L}(\mathscr{H})$, it is of the form

$$J_1^*(X) = \sum_{n=1}^{\infty} C_n^* X C_n$$

where $C_n \in \mathscr{L}(\mathscr{H})$ and $\sum C_n^* C_n$ is strongly convergent. Therefore

$$J_1(\rho) = \sum B_n(\pi\rho)\, B_n^* ,$$

where

$$B_n = C_n(Y - 1)$$

are linear operators from Dom Y into $\mathscr{H}$. Since T_t is a contraction semigroup on $\mathscr{T}(\mathscr{H})$,

$$\mathrm{tr}[W(\rho)] \leqslant 0$$

for all $\rho \in \mathscr{D}$. Putting $\rho = |\phi\rangle\langle\phi|$ where $\phi \in \mathrm{Dom}(Y)$ and using Eq. (3.4) yields Eq. (3.7). Finally by Eqs. (1.2) and (1.5)

$$0 = \sum B_n \,|\,\Omega\rangle\langle\Omega\,|\, B_n^*$$

so $B_n\Omega = 0$ for all n.

4. The Extension Property

All our results so far depend on the hypothesis that T_t possesses a pure stationary state $|\,\Omega\rangle\langle\Omega\,|$. In this section we consider a related extension property.

III. We say the dynamical semigroup T_t on $\mathcal{T}(\mathcal{H})$ has the extension property if for every dynamical semigroup T_t'' on $\mathcal{T}(\mathcal{H}'')$ where $\dim \mathcal{H}'' < \infty$ there exists a dynamical semigroup T_t' on $\mathcal{T}(\mathcal{H}')$, where $\mathcal{H}' = \mathcal{H} \oplus \mathcal{H}''$, such that

$$T_t' \begin{pmatrix} \rho & 0 \\ 0 & \rho'' \end{pmatrix} = \begin{pmatrix} T_t\rho & 0 \\ 0 & T_t''\rho'' \end{pmatrix}$$

for all $t \geqslant 0$, $\rho \in \mathcal{T}(\mathcal{H})$, $\rho'' \in \mathcal{T}(\mathcal{H}'')$.

THEOREM 4.1. *If T_t has the extension property* III *and*

$$T_t^*\{\mathscr{C}(\mathcal{H})\} \subseteq \mathscr{C}(\mathcal{H}) \tag{4.1}$$

for all $t \geqslant 0$, then there is a strongly continuous one-parameter contraction semigroup C_t on $\mathcal{H}$ with infinitesimal generator Y, and operators B_n : Dom $Y \to \mathcal{H}$, *such that*

$$\mathcal{D} = \{(Y-1)^{-1}\rho(Y^*-1)^{-1} : \rho \in \mathcal{T}(\mathcal{H})\} \tag{4.2}$$

is contained in the domain of the infinitesimal generator W of T_t. Moreover

$$\langle Y\phi, \phi\rangle + \langle \phi, Y\phi\rangle + \sum_{n=1}^{\infty} \langle B_n\phi, B_n\phi\rangle \leqslant 0 \tag{4.3}$$

for all $\phi \in$ Dom(Y) *and*

$$W(\rho) = Y\rho + \rho Y^* + \sum_{n=1}^{\infty} B_n\rho B_n^* \tag{4.4}$$

for all $\rho \in \mathcal{D}$.

Proof. We let $\mathcal{H}'' = \mathbb{C}$ and $T_t'' = 1$ for all $t \geqslant 0$. We suppose T_t' is a dynamical semigroup on $\mathcal{T}(\mathcal{H}')$ of the type given in hypothesis III. If $\Omega = 0 \oplus 1$, then $|\Omega\rangle\langle\Omega|$ is a pure stationary state for T_t'. From Eq. (4.1) and the positivity of T_t' one can deduce that

$$T_t'^*\{\mathscr{C}(\mathcal{H}')\} \subseteq \mathscr{C}(\mathcal{H}')$$

for all $t > 0$, so that Theorem 3.4 is applicable.

Since

$$C_t'\Omega = C_t'^*\Omega = \Omega$$

for all $t > 0$, it follows that

$$C_t'(\mathcal{H}) \subseteq \mathcal{H}, \qquad C_t'^*(\mathcal{H}) \subseteq \mathcal{H}$$

for all $t \geqslant 0$. Therefore Y', $(Y' - 1)^{-1}$, and B_n leave the subspace $\mathscr{H}$ invariant. This completes the proof.

It is rather interesting that Theorem 4.1 has a partial converse.

THEOREM 4.2. *Let T_t be a dynamical semigroup on $\mathscr{T}(\mathscr{H})$ with generator W and let C_t be a contraction semigroup on $\mathscr{H}$ with generator Y. Let $\mathscr{D}$ be given by Eq.* (4.2) *and let W be given by Eq.* (4.4) *for all $\rho \in \mathscr{D}$, where B_n are linear operators from* Dom Y *to $\mathscr{H}$ satisfying Eq.* (4.3). *If W is the closure of its restriction to $\mathscr{D}$, then T_t has the extension property* III.

Proof. Let T_t'' be a dynamical semigroup on $\mathscr{T}(\mathscr{H}'')$ where $\dim(\mathscr{H}'') < \infty$. By Lindblad's theorem [14] its generator W'' is of the form

$$W''(\rho) = Y''\rho + \rho Y''^* + \sum_{n=1}^{\infty} B_n''\rho B_n''^*.$$

If X, X'' are operators on $\mathscr{H}$, $\mathscr{H}''$, we denote by X' the operator $\left[\begin{smallmatrix} X & 0 \\ 0 & X'' \end{smallmatrix}\right]$ on $\mathscr{H}'$. We can then consider the evolution equation

$$\frac{d\rho}{dt} = W'\rho, \tag{4.5}$$

where

$$W'\rho = Y'\rho + \rho Y'^* + \sum_{n=1}^{\infty} B_n'\rho B_n'^*. \tag{4.6}$$

By [3] there is a dynamical semigroup T_t' on $\mathscr{T}(\mathscr{H}')$, called the minimal solution of Eq. (4.5), whose infinitesimal generator W' is given by Eq. (4.6) on the domain

$$\mathscr{D}' = \pi'\{\mathscr{T}(\mathscr{H}')\} \supseteq \left\{\begin{bmatrix} \pi\rho & 0 \\ 0 & \pi''\rho'' \end{bmatrix} : \rho \in \mathscr{T}(\mathscr{H}), \rho'' \in \mathscr{T}(\mathscr{H}'')\right\}.$$

Since $W'\rho = W\rho$ for all $\rho \in \mathscr{D}$ and W is the closure of its restriction to $\mathscr{D}$, W' is an extension of W. Therefore

$$T_t'\rho = T_t\rho$$

for all $\rho \in \mathscr{T}(\mathscr{H})$ and $t \geqslant 0$. Similarly

$$T_t'\rho'' = T_t''\rho''$$

for all $\rho \in \mathscr{T}(\mathscr{H}'')$ and $t \geqslant 0$.

5. Discussion

We note that Theorems 4.1 and 4.2 are not quite converses of each other since Theorem 4.1 requires the invariance of $\mathscr{C}(\mathscr{H})$ under T_t^*, while Theorem 4.2 requires that W be the closure of its restriction to $\mathscr{D}$.

Although the invariance of $\mathscr{C}(\mathscr{H})$ is needed in order to be able to apply Lemma 3.2, which is not true without some duality restrictions, we conjecture that Theorems 3.4 and 4.1 do not require this invariance assumption. In any case $\mathscr{C}(\mathscr{H})$ is frequently invariant for the following reason.

If T_t and V_t are dynamical semigroups on $\mathscr{T}(\mathscr{H})$ such that

$$\mathrm{tr}[(T_t\rho)\sigma] = \mathrm{tr}[\rho(V_t\sigma)]$$

for all $t \geqslant 0$ and $\rho, \sigma \in \mathscr{T}(\mathscr{H})$, then

$$T_t^*(\sigma) = V_t(\sigma)$$

so

$$T_t^*\{\mathscr{T}(\mathscr{H})\} \subseteq \mathscr{T}(\mathscr{H}).$$

Since $\mathscr{T}(\mathscr{H})$ is dense in $\mathscr{C}(\mathscr{H})$ for the operator norm and T_t^* is bounded,

$$T_t^*\{\mathscr{C}(\mathscr{H})\} \subseteq \mathscr{C}(\mathscr{H})$$

for all $t \geqslant 0$. Note that if T_t has generator

$$W(\rho) = Y\rho + \rho Y^* + \sum B_n \rho B_n^* ,$$

then the generator Z of V_t is formally

$$Z(\sigma) = Y^*\sigma + \sigma Y + \sum B_n^* \sigma B_n .$$

Concerning the condition that W be the closure of its restriction to $\mathscr{D}$ we note that there are examples where this is not the case [3]. It is shown in [3] that the evolution equation (4.5) then has infinitely many different solutions, corresponding to different reentry laws from infinity in the language of probability theory. A complete classification of the generators of dynamical semigroups must therefore contain something analogous to the boundary theory of Markov processes.

We finally comment that we do not know whether the extension property III is satisfied for all dynamical semigroups. If not, then one might try to justify it physically along the same lines as for complete positivity [12, 14].

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